Networks of Many Public Goods with Non-Linear Best Replies

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Abstract

We model a bipartite network where links connect agents and public goods. Agents play a voluntary contribution game, where they decide how much to contribute to each public good they are connected to. We show that the problem of finding a Nash equilibrium can be posed as a non-linear complementarity problem. The existence of an equilibrium point is established for a wide class of individual preferences. Then, we find a simple sufficient condition, on network structure only, that guarantees the uniqueness of equilibria. An easy procedure to build networks respecting this condition is finally provided.

Keywords: bipartite graph, public good, Nash equilibrium, non-linear complementarity problem.

JEL: C72, D85, H41.

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1 Introduction

In many social or geographic systems, multiple collective goods are produced voluntarily, simultaneously, and at different scales (Olson, 1965). Take for example an irrigated perimeter in which a farmer experiments simultaneously a new irrigation technology and a new crop variety. The information recolted from the new irrigation technology may be of common interest to a group of farmers, the results obtained from the new crop variety to another group. Similarly, consider a consumer that experiences several new products. Each experience may benefit a specific part of his familial and friendship relationships. Moreover, think about a municipality that introduces at the same time several resource conservation programs. Water conservation programs may spillover to municipalities within the same river basin, energy conservation programs to municipalities belonging to the same production site, soil conservation programs to direct neighboring municipalities, etc.

There exists no theoretic model analyzing the provision of many public goods in social or geographic networks as exemplified above. However, the principle is simple: n agents have to choose to contribute or not to m local public goods. But agents interact only with their " neighbors ", in other words, there are local network relationships between the agents, on each public good. Consideration of the structure of these relationships raises interesting problems. We focus on two important ones. How concave the utility functions should be to guarantee the existence of an equilibrium point? How equilibrium uniqueness is related to network structure?

To address these questions, we model a bipartite network where links connect agents with public goods.[1] We look at the voluntary contribution game, where agents decide how much to contribute to each public good they are connected to. The agents receive benefits from own and neighbors' contribution according to a concave benefit function.[2] The cost of contribution to each agent is a convex function of the total contribution from the agent.[3] Within this framework, we show that the problem of finding an equilibrium may be posed as a *non-linear complementarity problem* (Cottle, 1966; Kara-

[1]Bipartite networks have been used, for instance, to model economic exchange when buyers have relationships with sellers (Kranton and Minehart, 2001; Corominas-Bosch, 2004), labor market matching problems (Bóna, 2006), and the tragedy of the commons when the commons are multiple (Ilkiliç, 2011).

[2]The assumption of concave benefits is familiar in network games of one public good provision (Bloch and Zenginobuz, 2007; Bramoullé and Kranton, 2007; Corbo et al., 2007; Ballester and Calvó-Armengol, 2010; Bramoullé et al., 2014; Rébillé and Richefort, 2014). In general, this means that the value of the public good has a physical restriction.

[3]The assumption of convex costs suggests the presence of a private good and a budgetary constraint (see, e.g., Bergstrom et al., 1986; Bramoullé and Kranton, 2007).

mardian, 1969, 1972; Kolstad and Mathiesen, 1987).

This paper contributes to two main research areas. On one side, we study the voluntary and simultaneous provision of two or more public goods. Much work in this field has been concerned with neutrality problems (Kemp, 1984; Bergstrom et al., 1986; Cornes and Itaya, 2010), problems of equilibrium existence (Bergstrom et al., 1986; Cornes and Itaya, 2010) and efficiency problems (Cornes and Schweinberger, 1996; Cornes and Itaya, 2010). We extend the basic model of two or more public goods to a network of agents and public goods.[4] In other words, we consider a game of many public goods provision where the agents have multidimensional and heterogeneous strategy spaces. Given such a game, we show how the existence of a unique equilibrium is conditionned by the shape of individual preferences and the architecture of the network.

The other related line of literature is the analysis of network games with strategic substitutes. This class of games, pioneered among others by Ballester et al. (2006), encompasses various well-known games.[5] Under complete information[6], a uniqueness condition, that depends on network structure only, is established for the three following cases: linear best responses and unipartite network (Corbo et al., 2007; Ballester and Calvó-Armengol, 2010; Bramoullé et al., 2014), linear best responses and bipartite network (Ilkiliç, 2011), non-linear best responses and unipartite network (Rébillé and Richefort, 2014). Here, we study the fourth case, non-linear best responses and bipartite network, which generalizes the three other cases. Using techniques from the nonnegative matrix theory, we obtain a uniqueness condition that depends only on the structure of the graph.

In this paper, we are concerned with the existence and uniqueness of a pure-strategy Nash equilibrium (henceforth, PSNE) in a network game of many public goods provision. In Section 2, the voluntary contribution game is defined. In Section 3, the existence of a PSNE is established by requiring the appropriate shape in the individual preferences. In Section 4, it is shown that the voluntary contribution game admits a unique PSNE whenever the bipartite network is sufficiently sparse. In Section 5, we apply our results to

[4]The basic model of two or more public goods is a special case of our model when the network is complete and the substitutability between contributions is perfect.

[5]Network games of public good provision belong to the class of games of strategic substitutes and positive externalities (see, e.g., Bramoullé and Kranton, 2007; Galeotti et al., 2010). Network games of Cournot competition and network games of common property resources can be defined as games of strategic substitutes and negative externalities (see, e.g., Ilkiliç, 2011; Bramoullé et al., 2014).

[6]See Galeotti et al. (2010) for the analysis of network games under incomplete information.

networks in which the number of public goods equals the number of agents. Section 6 concludes. All the proofs are relegated at the Appendix.

2 A Model

Consider a model where there are m public goods $p_1, \ldots, p_m$ and n agents $a_1, \ldots, a_n$. They are embedded in a network that links agents and public goods. We will represent the network as a *bipartite graph*.[7]

An undirected bipartite graph $g = \langle P \cup A, L \rangle$ consists of a set of nodes formed by public goods $P = \{p_1, ..., p_m\}$ and agents $A = \{a_1, ..., a_n\}$, and a set of links L, each link joining an agent with a public good. A link between a_i and p_j will be denoted as ij.[8] We say that an agent a_i is *connected* to a public good p_j if there is a link between a_i and p_j. We will assume that an agent can choose to contribute or not to a public good if and only if he is connected to it. Let $r(g)$ be the number of links in L.

Given a graph g, we will denote by $N_g(p_j)$ the set of agents that are connected to p_j, i.e.,

$$N_g(p_j) = \{a_i \in A \text{ such that } ij \in L\},$$

and similarly, by $N_g(a_i)$ the set of public goods to which a_i is connected, i.e.,

$$N_g(a_i) = \{p_j \in P \text{ such that } ij \in L\}.$$

Then, $\sum_{a_i \in A} |N_g(a_i)| = \sum_{p_j \in P} |N_g(p_j)| = |L| = r(g)$. For all a_i, we note $r_i(g) = |N_g(a_i)|$ and for all p_j, $r^j(g) = |N_g(p_j)|$. We will assume, w.l.o.g., that each agent is connected to at least one public good and vice versa, i.e., $r_i(g)$ and $r^j(g)$ are in $\mathbb{N}^*$ for all a_i and for all p_j.[9]

Now we define the column vector that shows the contributions flowing at each link in L. Given a graph g, let $\mathbf{x}_g$ be the column vector of contributions.[10] Hence, $\mathbf{x}_g$ is the *link by link profile of contributions* and has size $r(g)$.

[7]Some of the basic notation introduced in this section is borrowed from Corominas-Bosch (2004) and Ilkiliç (2011).

[8]To avoid confusion, and since the network is undirected, we will respect the following rule in the notation of a link: the first small italic letter always refers to an agent and the second to a public good.

[9]In general, a public good is provided by at least two agents. Until Section 5, we will implicitely adopt this definition. However, since our results hold even if some public goods are provided by only one agent, we only need to impose that $r^j(g) \geq 1$ for all $p_j \in P$. See Section 5 for a discussion.

[10]All vectors considered in this paper are column vectors and are denoted by lowercase bold letters. We reserve the use of uppercase bold letters for matrices.

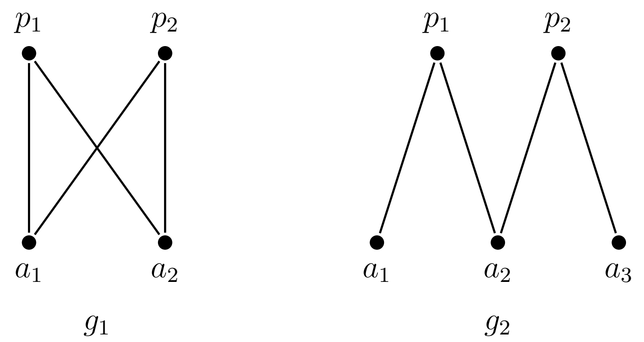

Figure 1: Networks with two public goods.

In the vector $\mathbf{x}_g$, the links are sorted in lexicographic order: the contribution x_{ij} is listed above the contribution x_{kl} when $i < k$ or when $i = k$ and $j < l$. For graphs g_1 and g_2 given above (Fig. 1),

$$\mathbf{x}_{g_1} = \begin{pmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{pmatrix} \quad \text{and} \quad \mathbf{x}_{g_2} = \begin{pmatrix} x_{11} \\ x_{21} \\ x_{22} \\ x_{32} \end{pmatrix}.$$

For a given graph g, the utility function of agent a_i is $U_i(\mathbf{x}_g)$. We will assume that the utility functions are additively separable into concave benefit and convex cost functions, all defined on $\mathbb{R}_+$ and continuously differentiable. For a given $\mathbf{x}_g \in \mathbb{R}_+^{r(g)}$,

$$U_i\left(\mathbf{x}_g\right) = \sum_{p_j \in N_g(a_i)} b_{ij}\left(x_{ij} + \sum_{a_k \in N_g(p_j)\setminus\{a_i\}} \lambda_{kj}^i x_{kj}\right) - c_i\left(\sum_{p_j \in N_g(a_i)} x_{ij}\right).$$

The first term is the (concave) benefit b_{ij} received from a public good p_j and summed over the public goods to which a_i is connected. The parameter $\lambda_{kj}^i \geq 0$ reflects the intensity of the positive externality received by agent a_i from agent a_k's contribution to public good p_j.[11] The second term is the (convex) cost c_i incurred by a_i. The utility function, although separable in terms of costs and benefits, is not separable with respect to each public good. In particular, the marginal utility from x_{ij} does depend on the contributions by a_i to public goods other than p_j. For example in graph g_1, the contribution by agent a_1 to public good p_1 depends on his contribution to the other public good p_2.

[11] Hence, λ_{kj}^i denotes the degree of substitutability between contribution x_{ij} and contribution x_{kj}, from the point of view of agent a_i (i.e., in general, $\lambda_{kj}^i \neq \lambda_{ij}^k$).

We consider the following *voluntary contribution game*. Given a graph g, each agent a_i maximizes his utility function with respect to x_{ij} constrained to be nonnegative for all $p_j \in N_g(a_i)$. So the set of players is the set of agents $A = \{a_1, \ldots, a_n\}$, and the strategy space of agent a_i is $(\mathbf{x}_g)_i \in \mathbb{R}_+^{r_i(g)}$. For a contribution profile $\mathbf{x}_g \in \mathbb{R}_+^{r(g)}$, each agent a_i earns payoffs $U_i(\mathbf{x}_g) \in \mathbb{R}$. We analyze the existence and the uniqueness of PSNE when individual decisions are simultaneous.

3 Equilibrium Existence

In the area of network games with strategic substitutes, the question of existence of a PSNE has received little attention. The reason is that individual preferences are generally specified such that best response functions are piecewise linear, whether the agents' strategy space is unidimensional (Bramoullé and Kranton, 2007; Ballester and Calvó-Armengol, 2010; Bramoullé et al., 2014) or multidimensional (Ilkiliç, 2011). In this section, the existence of a PSNE to a network game of strategic substitutes (i.e., the voluntary contribution game) is established when the strategy spaces of the agents are multidimensional and heterogeneous, and the set of best response functions define non-linear mappings.

Let μ_{ij} be the Karush-Kuhn-Tucker's multiplier associated with the constraint $x_{ij} \geq 0$. For all links $ij \in L$, the first order conditions are given by

$$b'_{ij}\left(x_{ij} + \sum_{a_k \in N_g(p_j)\setminus\{a_i\}} \lambda^i_{kj} x_{kj}\right) - c'_i\left(\sum_{p_j \in N_g(a_i)} x_{ij}\right) + \mu_{ij} = 0$$

with

$$\mu_{ij} x_{ij} = 0, \quad \mu_{ij} \geq 0.$$

We then deduce that all PSNE admitted by the voluntary contribution game are solutions to a non-linear complementarity problem.[12,13]

[12] Inequalities between vectors means inequalities between components. The superscript T denotes the transpose of a vector or a matrix.

[13] The complementarities in the network are between the contributions, which are either *strategic substitutes* or *complements*. For example in the complete graph g_1 (Fig. 1), x_{11} and x_{21} are *strategic substitutes*. They both participate to the provision of p_1. The contribution from one agent decreases the marginal benefit from p_1. This in turn decreases the incentive of the other agent to participate to the provision of p_1. Moreover, x_{21} and x_{22} are also *strategic substitutes*. They both come from a_2. The contribution to one public good increases the marginal cost incurred by a_2. This in turn decreases the incentive of a_2 to participate to the other public good. So x_{11} and x_{21}, as well as x_{21} and x_{22}, are *strategic substitutes*. This makes x_{11} and x_{22} *complements*.

Property 1. *Given a graph g, a profile $\mathbf{x}_g \in \mathbb{R}_+^{r(g)}$ is a PSNE of the voluntary contribution game if and only if $\mathbf{x}_g$ satisfies*

$$\mathbf{x}_g \geq \mathbf{0}, \quad \mathbf{b}'\left(\mathbf{D}_g\mathbf{x}_g\right) - \mathbf{c}'\left(\mathbf{M}_g\mathbf{x}_g\right) \leq \mathbf{0}, \quad {\mathbf{x}_g}^{\mathsf{T}}\left[\mathbf{b}'\left(\mathbf{D}_g\mathbf{x}_g\right) - \mathbf{c}'\left(\mathbf{M}_g\mathbf{x}_g\right)\right] = 0,$$

where for all links ij, $(\mathbf{b}'(\mathbf{D}_g\mathbf{x}_g))_{ij} = b'_{ij}(x_{ij} + \sum_{a_k \in N_g(p_j)\setminus\{a_i\}} \lambda^i_{kj} x_{kj})$ and $(\mathbf{c}'(\mathbf{M}_g\mathbf{x}_g))_{ij} = c'_i(\sum_{p_j \in N_g(a_i)} x_{ij})$.

For any graph g, the columns and the rows in $\mathbf{D}_g$ and $\mathbf{M}_g$ are the links in g. In both matrices, the links are classified in the same order as in $\mathbf{x}_g$: the rows (resp. the columns) are sorted such that the link ij is listed above (resp. to the left of) the link kl when $i < k$ or when $i = k$ and $j < l$. Then, $\mathbf{D}_g = [d_{ij,kl}]_{r(g)\times r(g)}$ is such that

$$d_{ij,kl} = \begin{cases} 1, & \text{for } ij, kl \in L \text{ s.t. } ij = kl; \\ \lambda^i_{kl}, & \text{for } ij, kl \in L \text{ s.t. } i \neq k \text{ and } j = l; \\ 0, & \text{for } ij, kl \in L \text{ s.t. } j \neq l. \end{cases}$$

We call $\mathbf{D}_g$ the *matrix of peer influences*. For example, let us take $\mathbf{D}_{g_1}$ and $\mathbf{D}_{g_2}$.

$$\mathbf{D}_{g_1} = \begin{pmatrix} 1 & 0 & \lambda^1_{21} & 0 \\ 0 & 1 & 0 & \lambda^1_{22} \\ \lambda^2_{11} & 0 & 1 & 0 \\ 0 & \lambda^2_{12} & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{D}_{g_2} = \begin{pmatrix} 1 & \lambda^1_{21} & 0 & 0 \\ \lambda^2_{11} & 1 & 0 & 0 \\ 0 & 0 & 1 & \lambda^2_{32} \\ 0 & 0 & \lambda^3_{22} & 1 \end{pmatrix}.$$

So $\mathbf{D}_g$ will generally be asymmetric, while $\mathbf{M}_g$ is symmetric by construction. Precisely, $\mathbf{M}_g = [m_{ij,kl}]_{r(g)\times r(g)}$ is such that

$$m_{ij,kl} = \begin{cases} 1, & \text{for } ij, kl \in L \text{ s.t. } i = k; \\ 0, & \text{for } ij, kl \in L \text{ s.t. } i \neq k. \end{cases}$$

We call $\mathbf{M}_g$ the *matrix of personal influences*. For example, let us take $\mathbf{M}_{g_1}$ and $\mathbf{M}_{g_2}$.

$$\mathbf{M}_{g_1} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{M}_{g_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The structure of any graph g is characterized by both $\mathbf{M}_g$ and $\mathbf{D}_g$. We will make use of these matrices in the next section, for the uniqueness problem. Now, for the existence of a solution to the voluntary contribution game, we have the following hypothesis.

Assumption 1 (Technical assumptions)**.**
(A1.1) For all links ij and for all agents a_i, $b'_{ij}(0) - c'_i(0) > 0$.
(A1.2) For all links ij and for all agents a_i, $b'_{ij}(\infty) - c'_i(\infty) < 0$.
(A1.3) For all links ij and for all agents a_i, b_{ij} and c_i are twice continuously differentiable, with b_{ij} strictly concave and c_i convex.

Reading Property 1 makes these technical assumptions very intuitive. If A1.1 is not satisfied, then agent a_i would not provide any contribution to public good p_j, and link ij could be ignored. If A1.2 is not satisfied, then agent a_i's optimization problem w.r.t. his contribution to public good p_j has no solution. A1.3 reflects the convexity of preferences. In other words, A1.1, A1.2 and A1.3 guarantee that each best response defines a continuous function from a compact and convex set to itself. Then, we can relie on Brouwer fixed-point theorem to establish the following result.

Theorem 1 (Existence Theorem)**.** *Given a graph g, the voluntary contribution game admits a PSNE whenever Assumption 1 is satisfied.*

This result generalizes Bergstrom et al. (1986)'s existence result to a network of agents and public goods. It also extends Rébillé and Richefort (2014)'s existence result to the multidimensional case. Furthermore, when the benefit and cost functions are quadratic (as, e.g., in Ilkiliç, 2011), it can be shown that the technical assumptions are always fulfilled.

Corollary 1. *Let the benefit function of link ij and the cost function of agent a_i be such that*

$$b_{ij}(x) = \alpha_{ij} x - \frac{\eta}{2} x^2 \quad \textit{and} \quad c_i(x) = \frac{\delta_i}{2} x^2$$

for $x \in \mathbb{R}_+$, where $\alpha_{ij}, \eta, \delta_i > 0$.[14] *Given a graph g, the voluntary contribution game always admits a PSNE.*

[14] In that case, a profile $\mathbf{x}_g \in \mathbb{R}_+^{r(g)}$ is a PSNE of the voluntary contribution game if and only if $\mathbf{x}_g$ satisfies

$$\mathbf{x}_g \geq \mathbf{0}, \quad \boldsymbol{\alpha}_g - (\eta \mathbf{D}_g + \mathbf{C}_g \mathbf{M}_g)\, \mathbf{x}_g \leq \mathbf{0}, \quad {\mathbf{x}_g}^{\mathsf{T}} \left[\boldsymbol{\alpha}_g - (\eta \mathbf{D}_g + \mathbf{C}_g \mathbf{M}_g)\, \mathbf{x}_g\right] = 0,$$

where $\boldsymbol{\alpha}_g = [\alpha_{ij}]_{1 \times r(g)}$ and $\mathbf{C}_g = [c_{ij,kl}]_{r(g) \times r(g)}$ is such that

$$c_{ij,kl} = \begin{cases} \delta_i, & \text{for } ij, kl \in L \text{ s.t. } ij = kl, \\ 0, & \text{for } ij, kl \in L \text{ s.t. } ij \neq kl. \end{cases}$$

In $\boldsymbol{\alpha}_g$, the links (i.e., the rows) are sorted as in $\mathbf{x}_g$. In $\mathbf{C}_g$, the links (i.e., the rows and the columns) are sorted as in $\mathbf{M}_g$ and $\mathbf{D}_g$. Ilkiliç (2011) studied a particular version of this problem, where $\alpha_{ij} = \alpha$ for all $ij \in L$, $\eta = 2\beta$, $\delta_i = \gamma$ for all $a_i \in A$ and $\lambda^i_{kj} = 1/2$ for all $ij, kj \in L$, $k \neq i$.

4 Equilibrium Uniqueness

Now, we establish a sufficient condition for a unique PSNE to the voluntary contribution game. This question has been studied intensively when there is only one public good. Several conditions have been established, whether best replies are linear (see, e.g., Bloch and Zenginobuz, 2007; Ballester and Calvó-Armengol, 2010; Bramoullé et al., 2014) or non-linear (see, e.g., Rébillé and Richefort, 2014). However, the more realistic case of several public goods has received much less attention. When there are two or more public goods and best replies are non-linear, we shall establish the uniqueness of equilibria with diagonally dominant matrices.

Definition 1 (Hadamard)**.** A real matrix $\mathbf{A} = [a_{ij}]_{n\times n}$ is said to be *row diagonally dominant* (rdd) if

$$|a_{ii}| \geq \sum_{j\neq i} |a_{ij}|, \quad i = 1, \ldots, n,$$

and *strictly row diagonally dominant* (srdd) if strict inequality holds for all i.

Square matrices with dominant diagonal play a paramount role in mathematical economics. This is essentially due to the fact that a srdd matrix with positive diagonal entries has all of its principal minors positive (Berman and Plemmons, 1994, Thm. 2.3 p. 134). Then, diagonally dominant matrices fall within the scope of the well-known Hawkins-Simon condition that guarantees the existence of a solution in the input-output system. They also serve as a basis for establishing the stability of a competitive market (see, e.g., McKenzie, 1960).[15]

With this definition in hand, we shall impose conditions on the structure of the network *only*, under which the voluntary contribution game admits at most one PSNE. By contrast with classic results on the uniqueness of solutions to the non-linear complementarity problem, we do not impose conditions on the mapping (here, the vector-valued function of marginal utilities).[16] We reason by contradiction and obtain the following result.

[15]In this literature, the usual definition of a diagonal dominant matrix is slightly more general than the one adopted in this paper (see McKenzie, 1960, p. 47).

[16]For instance, by contrast with Karamardian (1969), we do not assume that the vector-valued function of marginal utilities is (strictly) monotone. Moreover, unlike in Kolstad and Mathiesen (1987), we do not exclude the possibility that at PSNE, an agent may be just at the margin of choosing to contribute or not to a given public good. See Facchinei and Pang (2003) for a survey of sufficient conditions on the mapping for the uniqueness of solutions to the non-linear complementarity problem.

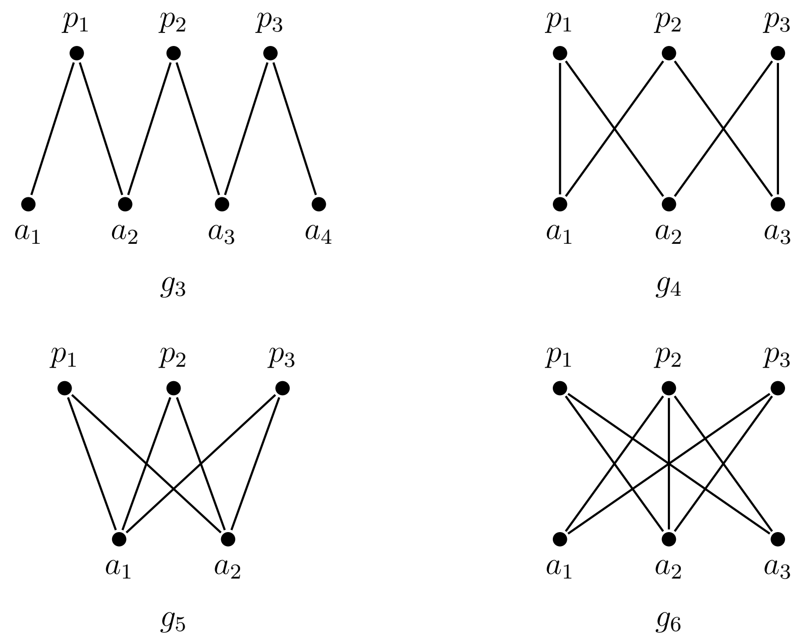

Figure 2: Networks with three public goods. Graphs g_3 and g_4 support the first condition of Theorem 2, while g_5 and g_6 do not.

Theorem 2 (Uniqueness Theorem)**.** *Let Assumption 1 be satisfied. Given a graph g, the voluntary contribution game admits a unique PSNE whenever*

$$r_i(g) \leq 2, \quad i = 1, \ldots, n,$$

and

$$\sum_{a_k \in N_g(p_j) \setminus \{a_i\}} \lambda_{kj}^i < 1$$

for all $ij \in L$.

For the proof, we show that the non-linear complementarity problem associated to the voluntary contribution game (see Property 1) admits at most one solution whenever $\mathbf{M}_g$ is rdd and $\mathbf{D}_g$ is srdd. Due to its Boolean nature, the matrix of personal influences $\mathbf{M}_g$ is rdd if and only if each agent is connected to at most two public goods, i.e.,

$$r_i(g) \leq 2, \quad i = 1, \ldots, n.$$

This does not mean, however, that there should be at most two public goods in the graph. It depends on the number of connections by agent. For example, in the four graphs given above (Fig. 2), there are three public goods but the

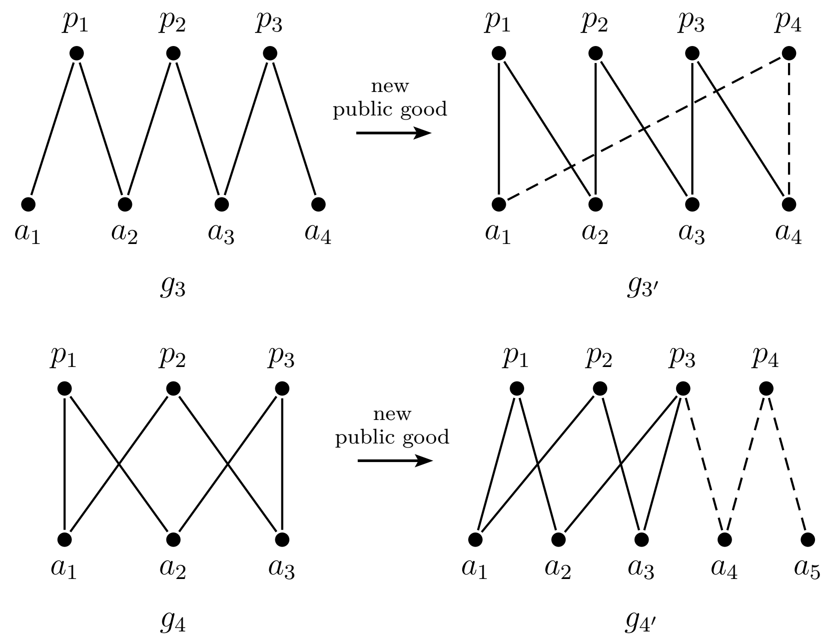

Figure 3: Adding public goods whilst respecting the first condition of Theorem 2.

structure of graphs g_3 and g_4 comply with the first assumption of Theorem 2 (i.e., $\mathbf{M}_{g_3}$ and $\mathbf{M}_{g_4}$ are rdd). By contrast, graphs g_5 and g_6 do not because each of these graphs contains at least one agent with three connections.

Furthermore, we can always add a new public good to a graph respecting the condition $r_i(g) \leq 2$. If there exists two agents with only one connection, the addition of a new public good can be done simply by creating a new connection from these agents towards the new public good. Otherwise, the addition of a new public good requires the introduction of new agents in the graph. Graphs $g_{3'}$ and $g_{4'}$ illustrate these two situations (Fig. 3).

The matrix of peer influences $\mathbf{D}_g$ is srdd if and only if each agent does not benefit too much from his peers, i.e.,

$$\sum_{a_k \in N_g(p_j) \setminus \{a_i\}} \lambda^i_{kj} < 1$$

for all $ij \in L$. Geometrically, these conditions on $\mathbf{M}_g$ and $\mathbf{D}_g$ mean that the voluntary contribution game admits a unique PSNE whenever the bipartite network is *sufficiently sparse*. If one agent is connected to three (or more) public goods[17] or if peer influences are too high, then the voluntary contri-

[17] This happens notably when the number of public goods is greater than the number of agents.

bution game might admit multiple PSNE. When there are only two public goods and peer influences are identical by public good, we have the following stronger results.

Corollary 2. *Let Assumption 1 be satisfied, and let* $\lambda^i_{kj} = \lambda_j$ *for all* $ij, kj \in L$, $k \neq i$.

(i) Given a graph g *where* $P = \{p_1, p_2\}$*, the voluntary contribution game admits a unique PSNE whenever*

$$\lambda_j < \frac{1}{r^j(g) - 1}, \quad j = 1, 2.$$

(ii) Given a complete graph g *where* $P = \{p_1, p_2\}$*, the voluntary contribution game admits a unique PSNE whenever*

$$\lambda_j < \frac{1}{n - 1}, \quad j = 1, 2.$$

For instance, graph g_2 falls within the scope of part (i) of Corollary 2, while part (ii) applies to graph g_1.

5 Application: the case $n = m$

Now we apply our results to networks in which the number of agents n equals the number of public goods m. We begin by specifying the nature of the public goods we consider in this section.

Definition 2. A public good p_j is a *collective good* if at least two agents participate to its provision, i.e., $r^j(g) \geq 2$. A public good p_j is an *individual good* if only one agent participates to its provision, i.e., $r^j(g) = 1$.

The structure of a graph $g = \langle P \cup A, L \rangle$ says which public good is collectively or individually provided. Let C and I denote the sets of collective and individual goods, respectively. Then, the set of public goods is the union of the sets of collective and individual goods, $P = C \cup I$. Let c denote the number of collective goods. Then $c = |C|$ and $0 \leq c \leq m$.

The first condition of Theorem 2 entails that, in a graph admitting a unique PSNE, no agent should have three or more connections, i.e., $r_i(g) \leq 2$ for all $a_i \in A$. Then necessarily, the number of public goods has an upper bound given by twice the number of agents, $m \leq 2n$. In particular, $m = 2n$ if and only if each agent is connected to two individual goods, that is $m = |I|$ or

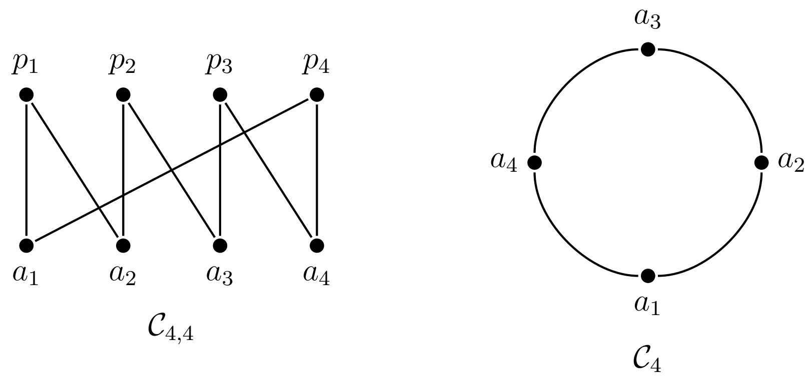

Figure 4: Circular graphs with four agents (and four public goods).

$c = 0$. In the same vein, the number of collective goods has an upper bound given by the number of agents, $c \leq n$. Indeed, for each collective good $p_j \in C$, we have $r^j(g) \geq 2$, so $r(g) = \sum_j r^j(g) \geq 2c$. Moreover, the number of links cannot exceed twice the number of agents, i.e., $r(g) = \sum_i r_i(g) \leq 2n$, since each agent has at most two connections. Hence $c \leq n$, in other words, the number of collective goods in a graph admitting a unique PSNE cannot exceed the number of agents.

Let us focus now on the case in which the number of collective goods equals the number of agents, $c = n$. So $r^j(g) = r_i(g) = 2$ for all p_j and for all a_i. A straightforward way to build a bipartite graph with $c = n$ is to consider the *circular bipartite graph* $\mathcal{C}_{n,n}$ or the *circular (unipartite) graph* $\mathcal{C}_n$ over the set of agents.

For $i = 1, \ldots, n-1$, agent a_i is connected to public goods p_i and p_{i+1} and agent a_n is connected to public goods p_n and p_1. So the collective good p_j is provided by agents a_j and a_{j-1} for $j = 2, \ldots, c$, and p_1 is provided by agents a_1 and a_n. Hence, any circular bipartite graph can be identified with a circular graph where nodes are identified with agents and links with public goods. For example, $g_{3'}$ (Fig. 3) is the $c = n = 4$ circular bipartite graph $\mathcal{C}_{4,4}$, which is identified with graph $\mathcal{C}_4$ (Fig. 4).

Circular bipartite graphs provide a simple procedure to build $c = n$ bipartite graphs. Let $\mathcal{C}_{n_1,n_1}, \ldots, \mathcal{C}_{n_K,n_K}$ be K circular bipartite graphs with $\mathcal{C}_{n_k,n_k} = \langle P_k \cup A_k, L_k \rangle$. Then, we may build a $c_1 + \ldots + c_K = n_1 + \ldots + n_K$ bipartite graph $\mathcal{C}_{n_1,n_1} + \ldots + \mathcal{C}_{n_K,n_K} = \langle P \cup A, L \rangle$ by *disjoint union* forming with $P = \cup_{k=1}^K P_k$, $A = \cup_{k=1}^K A_k$ and $L = \cup_{k=1}^K L_k$. For instance, two of the possible representations of the $c = n = 6$ bipartite graph are given by $\mathcal{C}_{6,6}$ and $\mathcal{C}_{4,4} + \mathcal{C}_{2,2}$ (see Fig. 5 below).

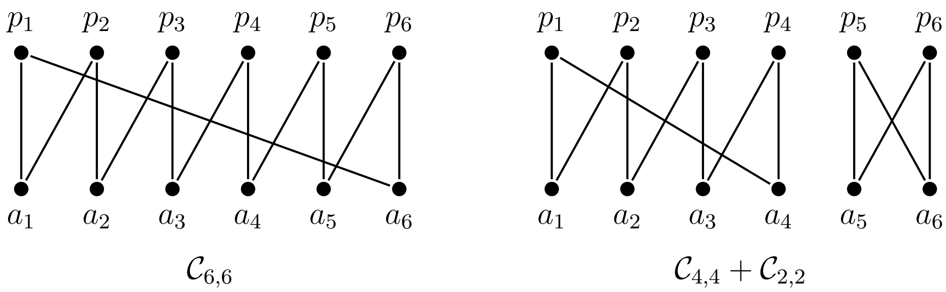

Figure 5: Two representations of the $c = n = 6$ bipartite graph.

The converse holds, i.e., any $c = n$ bipartite graph is a disjoint union of circular bipartite graphs. That is, for any $c = n$ bipartite graph g, there are circular bipartite graphs $\mathcal{C}_{n_1,n_1}, \ldots, \mathcal{C}_{n_K,n_K}$ with $\sum_{k=1}^{K} n_k = n$ and $n_k \geq 2$ for all k such that $g = \mathcal{C}_{n_1,n_1} + \ldots + \mathcal{C}_{n_K,n_K}$. This result can be formalized as follows.

Theorem 3 (Decomposition Theorem)**.** *Let $g = \langle P \cup A, L \rangle$ be a graph. Then, g is a $c = n$ bipartite graph if and only if there exists circular bipartite graphs $\mathcal{C}_{n_1,n_1}, \ldots, \mathcal{C}_{n_K,n_K}$ such that*

$$g = \mathcal{C}_{n_1,n_1} + \ldots + \mathcal{C}_{n_K,n_K}.$$

Moreover, the decomposition is unique.

Nevertheless, for given $c = n$, there exists different possible decompositions into several circular bipartite graphs. For instance, a $6 = 6$ bipartite graph can be obtained through $\mathcal{C}_{6,6}$, $\mathcal{C}_{4,4} + \mathcal{C}_{2,2}$, $\mathcal{C}_{3,3} + \mathcal{C}_{3,3}$ or $\mathcal{C}_{2,2} + \mathcal{C}_{2,2} + \mathcal{C}_{2,2}$. This question is intimately related to the *partition of an integer.*[18]

Any integer n can be partitioned into sums of integers. Let $p(n)$ be the number of (unordered) partitions of n. Formally,

$$p(n) = |\{(n_1, \ldots, n_K) : n = n_1 + \ldots + n_K, n_1 \geq \ldots \geq n_K \geq 1, n_k \in \mathbb{N}\}|,$$

for $n \geq 1$. Similarly, the number of (unordered) decompositions of a circular $c = n$ bipartite graph is given by $p_2(n)$, the number of (unordered) partitions of n with classes of size at least 2,

$$p_2(n) = |\{(n_1, \ldots, n_K) : n = n_1 + \ldots + n_K, n_1 \geq \ldots \geq n_K \geq 2, n_k \in \mathbb{N}\}|,$$

[18]See, e.g., Chapter 5 in Bóna (2006) for an introduction to this problem.

for $n \geq 2$. The connection is made through $p_2(n) = p(n) - p(n-1)$, that is $p(n) = p(n-1) + p_2(n)$. Indeed, a partition of n includes a class of size 1 or not. If the partition includes a class of size 1, then the partition without this class of size 1 is a partition of $n-1$. Otherwise, there is no class of size 1 in the partition, so each class is of size at least 2. Alternatively, we have

$$\begin{aligned} p(n) &= (p(n) - p(n-1)) + \ldots + (p(2) - p(1)) + p(1) \\ &= p_2(n) + \ldots + p_2(2) + 1. \end{aligned}$$

Any partition of n may contain $l \leq n-2$ classes of size 1, thus $n-l$ remains to be shared into classes of size at least 2, hence $p_2(n-l)$ possible partitions. Otherwise, the partition contains at least $n-1$ classes of size 1, in which case it coincides with the trivial partition into n classes of size 1. The following table illustrates this result when $c = n \leq 6$.

Table 1: Decompositions of $c = n \leq 6$ bipartite graphs, and the 11 partitions of 6.

n	Decompositions of $c = n$ bipartite graphs	Partitions of 6
6	$\mathcal{C}_{6,6}$	6
	$\mathcal{C}_{4,4} + \mathcal{C}_{2,2}$	4 + 2
	$\mathcal{C}_{3,3} + \mathcal{C}_{3,3}$	3 + 3
	$\mathcal{C}_{2,2} + \mathcal{C}_{2,2} + \mathcal{C}_{2,2}$	2 + 2 + 2
5	$\mathcal{C}_{5,5}$	1 + 5
	$\mathcal{C}_{3,3} + \mathcal{C}_{2,2}$	1 + 3 + 2
4	$\mathcal{C}_{4,4}$	1 + 1 + 4
	$\mathcal{C}_{2,2} + \mathcal{C}_{2,2}$	1 + 1 + 2 + 2
3	$\mathcal{C}_{3,3}$	1 + 1 + 1 + 3
2	$\mathcal{C}_{2,2}$	1 + 1 + 1 + 1 + 2
1		1 + 1 + 1 + 1 + 1 + 1

6 Conclusion

We have analyzed a network game of public good provision when there are many public goods. Under as weak as possible conditions on individual preferences, we show that there exists a unique PSNE whenever the bipartite network is sufficiently sparse. A simple procedure to build networks respecting the uniqueness condition is finally established for graphs in which the number of agents equals the number of public goods.

These results have been derived for the (general) case of network games with non-linear best replies and multidimensional strategy spaces. To our knwoledge, all previous results on equilibrium existence for network games of one public good provision are special cases of our existence result. We believe, however, that the main contribution of the paper is Theorem 2, since this result is the first to provide a sufficient condition, that depends on network structure only, for the uniqueness of equilibria in network games of many local public goods provision. Interestingly, it applies to all games that can be studied through the same complementarity problem as the one described by Property 1. That is the case, for instance, for network games of strategic substitutes and negative externalities such as the game of Cournot competition or the water extraction game (see, e.g., Okuguchi, 1983; Kolstad and Mathiesen, 1987; Ilkiliç, 2011).

Our analysis paves way for further research. Firstly, it should be explored if a sharper condition for uniqueness can be obtained. In particular, does the P-matrix condition established when there is only one public good hold when there are two or more public goods? Answering this question might require the use of other algebra techniques as, e.g., determinental inequalities for product and sum of matrices. Secondly, since we know when the equilibrium exists and is unique, it may be possible to study the structure of the equilibrium. When best replies are linear, numerous works in the branch of network games have expressed the equilibrium in terms of the Katz-Bonacich centrality vector (see, e.g., Ballester et al., 2006; Ballester and Calvó-Armengol, 2010). Although when best replies are non-linear, the relationship between the equilibrium and the Katz-Bonacich centrality vector is less obvious, this question remains an important challenge. Then, it may be interesting to implement our model in an experiment, in order to test if behaviors conform well to theoretical predictions.

Appendix

Proof of Theorem 1. Since $b_{ij} - c_i$ is strictly concave, $b'_{ij}(0) - c'_i(0) > 0$ and $b'_{ij}(M) - c'_i(M) < 0$ for some $M > 0$, there exists a unique x^*_{ij} such that $b'_{ij}(x^*_{ij}) - c'_i(x^*_{ij}) = 0$. Moreover, x^*_{ij} is link ij's maximum.

Let $S_{-i,j} = \sum_{a_k \in N_g(p_j) \setminus \{a_i\}} \lambda^i_{kj} x_{kj} \geq 0$ and $C_{i,-j} = \sum_{p_l \in N_g(a_i) \setminus \{p_j\}} x_{il} \geq 0$. Agent a_i's utility is given by

$$U_i(\mathbf{x}_g) = \sum_{p_j \in N_g(a_i)} b_{ij}(x_{ij} + S_{-i,j}) - c_i(x_{ij} + C_{i,-j})$$

By assumption, b_{ij} is strictly concave and c_i is convex, so $b'_{ij} - c'_i$ is strictly decreasing and continuous. Given $S_{-i,j}$ and $C_{i,-j}$, the best response for each link $ij \in L$ is

$$\phi_{ij}(S_{-i,j}, C_{i,-j}) = \begin{cases} \left[b'_{ij}(. + S_{-i,j}) - c'_i(. + C_{i,-j})\right]^{-1}(0), & \\ & \text{if } b'_{ij}(0 + S_{-i,j}) - c'_i(0 + C_{i,-j}) > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Since $b'_{ij}(. + S_{-i,j}) \leq b'_{ij}(.)$ and $c'_i(. + C_{i,-j}) \geq c'_i(.)$, we have

$$b'_{ij}(. + S_{-i,j}) - c'_i(. + C_{i,-j}) \leq b'_{ij}(.) - c'_i(.),$$

so

$$\phi_{ij}(S_{-i,j}, C_{i,-j}) \leq \phi_{ij}(0,0) = x^*_{ij}.$$

It follows that the autarkic contribution is always greater than the equilibrium contribution in a bipartite network.

Let us check that the best response is continuous w.r.t. $S_{-i,j}$ and $C_{i,-j}$. Let $S_{-i,j}, C_{i,-j} \geq 0$.

1st case: $b'_{ij}(0 + S_{-i,j}) - c'_i(0 + C_{i,-j}) < 0$. Then, $\phi_{ij}(S_{-i,j}, C_{i,-j}) = 0$. Since b'_{ij} and c'_i are continuous, there exists some neighborhood V of $S_{-i,j}$ and W of $C_{i,-j}$ such that $b'_{ij}(S) - c'_i(C) < 0$ for $S \in V$ and $C \in W$. Thus, $\phi_{ij}(S, C) = 0$ for $S \in V$ and $C \in W$, so ϕ_{ij} is continuous at $S_{-i,j}$ and $C_{i,-j}$.

2nd case: $b'_{ij}(0 + S_{-i,j}) - c'_i(0 + C_{i,-j}) \geq 0$. By definition, $(x_{ij}, S_{-i,j}, C_{i,-j})$ with $x_{ij} = \phi_{ij}(S_{-i,j}, C_{i,-j})$ is a solution to the equation

$$z_{ij}(x, S, C) = b'_{ij}(x + S) - c'_i(x + C) = 0.$$

Now, observe that

$$\frac{\partial z_{ij}}{\partial x}(x_{ij}, S_{-i,j}, C_{i,-j}) = b''_{ij}(x_{ij} + S_{-i,j}) - c''_i(x_{ij} + C_{i,-j}) < 0$$

by strict-concavity, so in accordance with the implicit function theorem, there exists some differentiable function ζ such that

$$\zeta\left(S_{-i,j}, C_{i,-j}\right) = x_{ij}$$

on some open neighborhood V of $(S_{-i,j}, C_{i,-j})$, satisfying

$$z_{ij}\left(\zeta(S,C), S, C\right) = b'_{ij}\left(\zeta(S,C) + S\right) - c'_i\left(\zeta(S,C) + C\right) = 0.$$

Thus, $\phi_{ij}(S_{-i,j}, C_{i,-j}) = x_{ij} = \zeta\left(S_{-i,j}, C_{i,-j}\right)$ on $V \ni (S_{-i,j}, C_{i,-j})$, so ϕ_{ij} is continuous at $S_{-i,j}$ and $C_{i,-j}$.

Consider the mapping

$$\begin{aligned} \Phi : \quad \prod_{ij \in L} \left[0, x^*_{ij}\right] \quad &\to \quad \prod_{ij \in L} \left[0, x^*_{ij}\right] \\ \mathbf{x}_g \quad &\mapsto \quad \left(\phi_{ij} \left(\sum_{a_k \in N_g(p_j) \setminus \{a_i\}} \lambda^i_{kj} x_{kj}, \sum_{p_l \in N_g(a_i) \setminus \{p_j\}} x_{il} \right) \right)_{ij} \end{aligned}$$

Φ is continuous w.r.t. $\mathbf{x}_g$ since ϕ_{ij}, $\mathbf{x}_g \mapsto \sum_{a_k \in N_g(p_j) \setminus \{a_i\}} \lambda^i_{kj} x_{kj}$ and $\mathbf{x}_g \mapsto \sum_{p_l \in N_g(a_i) \setminus \{p_j\}} x_{il}$ are continuous for all ij. According to Brouwer fixed-point theorem, Φ admits a fixed-point $\mathbf{x}_g$ which is a PSNE of the voluntary contribution game, by construction. □

The following lemma plays an important role in establishing our uniqueness result.

Lemma 1. *Let g be a graph. For all $\mathbf{x}^1_g, \mathbf{x}^2_g$ in $\mathbb{R}^{r(g)}_+$ with $\mathbf{x}^1_g \neq \mathbf{x}^2_g$, there exists a link ij such that*

$$\left(x^1_{ij} - x^2_{ij}\right)\left[\frac{\partial U_i}{\partial x_{ij}}\left(\mathbf{x}^1_g\right) - \frac{\partial U_i}{\partial x_{ij}}\left(\mathbf{x}^2_g\right)\right] < 0$$

whenever $\mathbf{M}_g$ is rdd and $\mathbf{D}_g$ is srdd.

For its proof, we need to remind the class of P-matrices.

Definition 3 (Fiedler and Pták, 1962)**.** An $n \times n$ real matrix $\mathbf{A}$ is said to be a *P-matrix* if there exists k such that $x_k(\mathbf{A}\mathbf{x})_k > 0$ for all nonzero $\mathbf{x}$ in $\mathbb{R}^n$.

A srdd matrix with positive diagonal entries is a P-matrix (see Berman and Plemmons, 1994, Thm. 2.3 p. 134, M_{35} implies A_5).

Proof of Lemma 1. Let $\mathbf{x}_g^1$ and $\mathbf{x}_g^2$ be two arbitrary vectors in $\mathbb{R}_+^{r(g)}$. For each link ij, let

$$\psi_{ij}(\varepsilon) = \frac{\partial U_i}{\partial x_{ij}} \left(\varepsilon \mathbf{x}_g^1 + (1-\varepsilon) \mathbf{x}_g^2 \right).$$

Since $\mathbb{R}_+^{r(g)}$ is convex, $\varepsilon \mathbf{x}_g^1 + (1-\varepsilon)\mathbf{x}_g^2 \in \mathbb{R}_+^{r(g)}$ for all $0 \leq \varepsilon \leq 1$. We have

$$\psi_{ij}(1) - \psi_{ij}(0) = \frac{\partial U_i}{\partial x_{ij}} \left(\mathbf{x}_g^1 \right) - \frac{\partial U_i}{\partial x_{ij}} \left(\mathbf{x}_g^2 \right),$$

$$\psi'_{ij}(\varepsilon) = \nabla \frac{\partial U_i}{\partial x_{ij}} (\mathbf{x}_g) \left(\mathbf{x}_g^1 - \mathbf{x}_g^2 \right),$$

where $\mathbf{x}_g = \varepsilon \mathbf{x}_g^1 + (1-\varepsilon)\mathbf{x}_g^2 \in \mathbb{R}_+^{r(g)}$ and $\nabla \frac{\partial U_i}{\partial x_{ij}}(\mathbf{x}_g)$ is the gradient of $\frac{\partial U_i}{\partial x_{ij}}$ at $\mathbf{x}_g$. Then, $\nabla \frac{\partial U_i}{\partial x_{ij}}(\mathbf{x}_g)$ is a row vector of size $r(g)$, in which the columns (i.e., the links) are sorted as in (the transpose of) $\mathbf{x}_g$. Applying the mean-value theorem on ψ_{ij}, we have

$$\psi_{ij}(1) - \psi_{ij}(0) = \psi'_{ij}(\bar{\varepsilon}_{ij}) = \nabla \frac{\partial U_i}{\partial x_{ij}} \left(\bar{\mathbf{x}}_g^{[ij]} \right) \left(\mathbf{x}_g^1 - \mathbf{x}_g^2 \right)$$

for some $0 < \bar{\varepsilon}_{ij} < 1$, where $\bar{\mathbf{x}}_g^{[ij]} = \bar{\varepsilon}_{ij}\mathbf{x}_g^1 + (1-\bar{\varepsilon}_{ij})\mathbf{x}_g^2 \in \mathbb{R}_+^{r(g)}$. Thus, for each link ij,

$$\frac{\partial U_i}{\partial x_{ij}} \left(\mathbf{x}_g^1 \right) - \frac{\partial U_i}{\partial x_{ij}} \left(\mathbf{x}_g^2 \right) = \left(\mathbf{J}_{U'}(\bar{\mathbf{x}}_g) \left(\mathbf{x}_g^1 - \mathbf{x}_g^2 \right) \right)_{ij}$$

where $\mathbf{J}_{U'}(\bar{\mathbf{x}}_g)$ is the $r(g) \times r(g)$ " Jacobian " matrix[19] of the marginal utilities, where the rows and the columns (i.e., the links) are sorted as in $\mathbf{M}_g$ and $\mathbf{D}_g$. Then, $\mathbf{J}_{U'}(\bar{\mathbf{x}}_g)$ is such that each row ij is given by the gradient $\nabla \frac{\partial U_i}{\partial x_{ij}}(\bar{\mathbf{x}}_g^{[ij]})$.[20]

[19] This is not the " true " Jacobian matrix, since the Jacobian matrix $\mathbf{J}_F(\mathbf{x})$ of a differentiable mapping $F : D \to \mathbb{R}^n$, where D is a closed rectangular region of $\mathbb{R}^n$, is evaluated at a given $\mathbf{x} \in D$.

[20] For example, let us take $\mathbf{J}_{U'}(\bar{\mathbf{x}}_{g_1})$ (cf. graph g_1 at Fig. 1).

$$\mathbf{J}_{U'}(\bar{\mathbf{x}}_{g_1}) = \begin{pmatrix} \nabla \frac{\partial U_1}{\partial x_{11}} \left(\bar{\mathbf{x}}_{g_1}^{[11]} \right) \\ \nabla \frac{\partial U_1}{\partial x_{12}} \left(\bar{\mathbf{x}}_{g_1}^{[12]} \right) \\ \nabla \frac{\partial U_2}{\partial x_{21}} \left(\bar{\mathbf{x}}_{g_1}^{[21]} \right) \\ \nabla \frac{\partial U_2}{\partial x_{22}} \left(\bar{\mathbf{x}}_{g_1}^{[22]} \right) \end{pmatrix} = \begin{pmatrix} \nabla \frac{\partial U_1}{\partial x_{11}} \left(\bar{\varepsilon}_{11}\mathbf{x}_{g_1}^1 + (1-\bar{\varepsilon}_{11})\mathbf{x}_{g_1}^2 \right) \\ \nabla \frac{\partial U_1}{\partial x_{12}} \left(\bar{\varepsilon}_{12}\mathbf{x}_{g_1}^1 + (1-\bar{\varepsilon}_{12})\mathbf{x}_{g_1}^2 \right) \\ \nabla \frac{\partial U_2}{\partial x_{21}} \left(\bar{\varepsilon}_{21}\mathbf{x}_{g_1}^1 + (1-\bar{\varepsilon}_{21})\mathbf{x}_{g_1}^2 \right) \\ \nabla \frac{\partial U_2}{\partial x_{22}} \left(\bar{\varepsilon}_{22}\mathbf{x}_{g_1}^1 + (1-\bar{\varepsilon}_{22})\mathbf{x}_{g_1}^2 \right) \end{pmatrix}.$$

Now, given $\frac{\partial U_i}{\partial x_{ij}}(\bar{\mathbf{x}}_g^{[ij]})$ where $ij \in L$, observe that

$$\frac{\partial^2 U_i}{\partial x_{kl}\partial x_{ij}}\left(\bar{\mathbf{x}}_g^{[ij]}\right) = \begin{cases} b''_{ij}\left(\bar{x}_{ij}^{[ij]} + \sum_{a_k \in N_g(p_j)\setminus\{a_i\}} \lambda_{kj}^i \bar{x}_{kj}^{[ij]}\right) - c''_i\left(\sum_{p_j \in N_g(a_i)} \bar{x}_{ij}^{[ij]}\right), & \\ \qquad \text{for } kl \in L \text{ s.t. } kl = ij; & \\ -c''_i\left(\sum_{p_j \in N_g(a_i)} \bar{x}_{ij}^{[ij]}\right), \quad \text{for } kl \in L \text{ s.t. } k = i \text{ and } l \neq j; & \\ \lambda_{kj}^i b''_{ij}\left(\bar{x}_{ij}^{[ij]} + \sum_{a_k \in N_g(p_j)\setminus\{a_i\}} \lambda_{kj}^i \bar{x}_{kj}^{[ij]}\right), & \\ \qquad \text{for } kl \in L \text{ s.t. } k \neq i \text{ and } l = j; & \\ 0, \qquad \text{for } kl \in L \text{ s.t. } k \neq i \text{ and } l \neq j. & \end{cases}$$

Hence,

$$\mathbf{J}_{U'}\left(\bar{\mathbf{x}}_g\right) = \mathtt{B}\left(\bar{\mathbf{x}}_g\right)\mathbf{D}_g - \mathtt{C}\left(\bar{\mathbf{x}}_g\right)\mathbf{M}_g \iff -\mathbf{J}_{U'}\left(\bar{\mathbf{x}}_g\right) = \mathtt{C}\left(\bar{\mathbf{x}}_g\right)\mathbf{M}_g - \mathtt{B}\left(\bar{\mathbf{x}}_g\right)\mathbf{D}_g$$

where $\mathtt{B}\left(\bar{\mathbf{x}}_g\right) = [\mathtt{b}_{ij,kl}]_{r(g)\times r(g)}$ is such that[21]

$$\mathtt{b}_{ij,kl} = \begin{cases} b''_{ij}\left(\bar{x}_{ij}^{[ij]} + \sum_{a_k \in N_g(p_j)\setminus\{a_i\}} \lambda_{kj}^i \bar{x}_{kj}^{[ij]}\right), & \text{for } ij, kl \in L \text{ s.t. } ij = kl; \\ 0, & \text{for } ij, kl \in L \text{ s.t. } ij \neq kl; \end{cases}$$

and $\mathtt{C}(\bar{\mathbf{x}}_g) = [\mathtt{c}_{ij,kl}]_{r(g)\times r(g)}$ is such that

$$\mathtt{c}_{ij,kl} = \begin{cases} c''_i\left(\sum_{p_j \in N_g(a_i)} \bar{x}_{ij}^{[ij]}\right), & \text{for } ij, kl \in L \text{ s.t. } ij = kl; \\ 0, & \text{for } ij, kl \in L \text{ s.t. } ij \neq kl. \end{cases}$$

By assumption, $\mathbf{D}_g$ is srdd. Then, so is $-\mathtt{B}(\bar{\mathbf{x}}_g)\mathbf{D}_g$ since $-\mathtt{B}(\bar{\mathbf{x}}_g)$ is a diagonal matrix with positive diagonal entries (by strict-concavity of the benefit functions). In addition, $\mathbf{M}_g$ is rdd. Then, so is $\mathtt{C}(\bar{\mathbf{x}}_g)\mathbf{M}_g$ since $\mathtt{C}(\bar{\mathbf{x}}_g)$ is a diagonal matrix with nonnegative diagonal entries (by convexity of the cost functions). Thus, $-\mathbf{J}_{U'}(\bar{\mathbf{x}}_g)$ is a P-matrix, since it is a srdd matrix with positive diagonal entries (Berman and Plemmons, 1994). By definition, there

[21] In both $\mathtt{B}(\bar{\mathbf{x}}_g)$ and $\mathtt{C}(\bar{\mathbf{x}}_g)$, the rows and the columns (i.e., the links) are sorted as in $\mathbf{M}_g$ and $\mathbf{D}_g$.

exists a link ij such that

$$\left(x_{ij}^1 - x_{ij}^2\right)\left(-\mathbf{J}_{U'}\left(\bar{\mathbf{x}}_g\right)\left(\mathbf{x}_g^1 - \mathbf{x}_g^2\right)\right)_{ij} > 0 \iff$$

$$\left(x_{ij}^1 - x_{ij}^2\right)\left(\mathbf{J}_{U'}\left(\bar{\mathbf{x}}_g\right)\left(\mathbf{x}_g^1 - \mathbf{x}_g^2\right)\right)_{ij} < 0,$$

thus,

$$\left(x_{ij}^1 - x_{ij}^2\right)\left[\frac{\partial U_i}{\partial x_{ij}}\left(\mathbf{x}_g^1\right) - \frac{\partial U_i}{\partial x_{ij}}\left(\mathbf{x}_g^2\right)\right] < 0.$$

□

We are now ready to prove Theorem 2.

Proof of Theorem 2. Let us assume that there are two PSNE, $\mathbf{x}_g^1 \neq \mathbf{x}_g^2$. In accordance with Property 1, for each link ij,

$$x_{ij}^\alpha\left[b'_{ij}\left(x_{ij}^\alpha + \sum_{a_k \in N_g(p_j)\setminus\{a_i\}} \lambda_{kj}^i x_{kj}^\alpha\right) - c'_i\left(\sum_{p_j \in N_g(a_i)} x_{ij}^\alpha\right)\right] = 0, \quad \alpha = 1, 2,$$

and

$$b'_{ij}\left(x_{ij}^\alpha + \sum_{a_k \in N_g(p_j)\setminus\{a_i\}} \lambda_{kj}^i x_{kj}^\alpha\right) - c'_i\left(\sum_{p_j \in N_g(a_i)} x_{ij}^\alpha\right) \leq 0, \quad \alpha = 1, 2.$$

Since $\mathbf{x}_g^1, \mathbf{x}_g^2 \geq \mathbf{0}$, for each link ij, it holds

$$x_{ij}^1\left[b'_{ij}\left(x_{ij}^2 + \sum_{a_k \in N_g(p_j)\setminus\{a_i\}} \lambda_{kj}^i x_{kj}^2\right) - c'_i\left(\sum_{p_j \in N_g(a_i)} x_{ij}^2\right)\right] \leq 0$$

and

$$x_{ij}^2\left[b'_{ij}\left(x_{ij}^1 + \sum_{a_k \in N_g(p_j)\setminus\{a_i\}} \lambda_{kj}^i x_{kj}^1\right) - c'_i\left(\sum_{p_j \in N_g(a_i)} x_{ij}^1\right)\right] \leq 0.$$

It follows that, for each link ij,

$$\begin{aligned}
& x_{ij}^1\left[b'_{ij}\left(x_{ij}^2 + \sum_{a_k \in N_g(p_j)\setminus\{a_i\}} \lambda_{kj}^i x_{kj}^2\right) - c'_i\left(\sum_{p_j \in N_g(a_i)} x_{ij}^2\right)\right] \\
+ \; & x_{ij}^2\left[b'_{ij}\left(x_{ij}^1 + \sum_{a_k \in N_g(p_j)\setminus\{a_i\}} \lambda_{kj}^i x_{kj}^1\right) - c'_i\left(\sum_{p_j \in N_g(a_i)} x_{ij}^1\right)\right] \\
- \; & x_{ij}^1\left[b'_{ij}\left(x_{ij}^1 + \sum_{a_k \in N_g(p_j)\setminus\{a_i\}} \lambda_{kj}^i x_{kj}^1\right) - c'_i\left(\sum_{p_j \in N_g(a_i)} x_{ij}^1\right)\right] \\
- \; & x_{ij}^2\left[b'_{ij}\left(x_{ij}^2 + \sum_{a_k \in N_g(p_j)\setminus\{a_i\}} \lambda_{kj}^i x_{kj}^2\right) - c'_i\left(\sum_{p_j \in N_g(a_i)} x_{ij}^2\right)\right] \leq 0,
\end{aligned}$$

thus,

$$\boxed{\mathbf{A}}:\ \left(x_{ij}^1 - x_{ij}^2\right)\left[b'_{ij}\left(x_{ij}^2 + \sum_{a_k\in N_g(p_j)\backslash\{a_i\}} \lambda_{kj}^i x_{kj}^2\right) - c'_i\left(\sum_{p_j\in N_g(a_i)} x_{ij}^2\right)\right.$$
$$\left. - b'_{ij}\left(x_{ij}^1 + \sum_{a_k\in N_g(p_j)\backslash\{a_i\}} \lambda_{kj}^i x_{kj}^1\right) + c'_i\left(\sum_{p_j\in N_g(a_i)} x_{ij}^1\right)\right] \leq 0.$$

Since $r_i(g) \leq 2$ for all $a_i \in A$, $\mathbf{M}_g$ is rdd. Moreover, $\mathbf{D}_g$ is srdd as $\sum_{a_k\in N_g(p_j)\backslash\{a_i\}} \lambda_{kj}^i < 1$ for all $ij \in L$. Then, according to Lemma 1, there exists a link ij such that

$$\left(x_{ij}^1 - x_{ij}^2\right)\left[\frac{\partial U_i}{\partial x_{ij}}\left(\mathbf{x}_g^1\right) - \frac{\partial U_i}{\partial x_{ij}}\left(\mathbf{x}_g^2\right)\right] < 0 \iff$$
$$\left(x_{ij}^1 - x_{ij}^2\right)\left[\frac{\partial U_i}{\partial x_{ij}}\left(\mathbf{x}_g^2\right) - \frac{\partial U_i}{\partial x_{ij}}\left(\mathbf{x}_g^1\right)\right] > 0,$$

or equivalently,

$$\left(x_{ij}^1 - x_{ij}^2\right)\left[b'_{ij}\left(x_{ij}^2 + \sum_{a_k\in N_g(p_j)\backslash\{a_i\}} \lambda_{kj}^i x_{kj}^2\right) - c'_i\left(\sum_{p_j\in N_g(a_i)} x_{ij}^2\right)\right.$$
$$\left. - b'_{ij}\left(x_{ij}^1 + \sum_{a_k\in N_g(p_j)\backslash\{a_i\}} \lambda_{kj}^i x_{kj}^1\right) + c'_i\left(\sum_{p_j\in N_g(a_i)} x_{ij}^1\right)\right] > 0,$$

contradicting $\boxed{\mathbf{A}}$. So, $\mathbf{x}_g^1 - \mathbf{x}_g^2 = \mathbf{0}$ and uniqueness is established. □

Proof of Theorem 3. Let us prove it by induction on N the number of agents. It is immediate to check that a $2 = 2$ or a $3 = 3$ bipartite graph is a circular bipartite graph $\mathcal{C}_{2,2}$ or $\mathcal{C}_{3,3}$.

Assume for $N \geq 3$, that any $c = n$ bipartite graph with $c \leq N$ can be decomposed into circular bipartite graphs. Let g be a $c = n = N+1$ bipartite graph. Let $(a_{i_1}, p_{j_1}) \in A\times P$ with $i_1 j_1 \in L$. There exists $p_{j_2} \in P$ with $j_2 \neq j_1$ such that $i_1 j_2 \in L$ (since $r_{i_1}(g) = 2$) and then, there exists some $a_{i_2} \in A$ with $i_2 \neq i_1$ such that $i_2 j_2 \in L$ (since $r^{j_2}(g) = 2$).

If $i_2 j_1 \in L$, then g admits a $2 = 2$ bipartite subgraph and g', the restriction of g to $A\backslash\{a_{i_1}, a_{i_2}\}$ and $P\backslash\{p_{j_1}, p_{j_2}\}$, remains a $c = n = N-1$ bipartite graph, so by induction hypothesis g' admits a decomposition into circular bipartite graphs $\mathcal{C}_{n_1,n_1}+\ldots+\mathcal{C}_{n_K,n_K}$. Thus, g is the disjoint union of $\mathcal{C}_{n_1,n_1}+\ldots+\mathcal{C}_{n_K,n_K}$ and $\mathcal{C}_{2,2}$.

Otherwise, $i_2 j_1 \notin L$. So there exists $p_{j_3} \in P$ with $j_3 \neq j_1, j_2$ such that $i_2 j_3 \in L$. Then, there exists some $a_{i_3} \in A$ with $i_3 \neq i_1, i_2$ (since $r_{i_1}(g), r_{i_2}(g) \leq 2$) such that $i_3 j_3 \in L$. Again, if $i_3 j_1 \in L$, g admits a $3 = 3$ circular bipartite subgraph and g'', the restriction of g to $A \backslash \{a_{i_1}, a_{i_2}, a_{i_3}\}$ and $P \backslash \{p_{j_1}, p_{j_2}, p_{j_3}\}$, remains a $c = n = N - 2$ bipartite graph. Thus, g is the disjoint union of $\mathcal{C}_{n_1,n_1} + \ldots + \mathcal{C}_{n_K,n_K}$ and $\mathcal{C}_{3,3}$.

Otherwise, $i_3 j_1 \notin L$, and so on. The process stops since we may extract at most $N+1$ public goods. In that case, the final link is $i_{N+1} j_1$. Hence, the $c = n = N+1$ bipartite graph is precisely $\mathcal{C}_{N+1,N+1}$. □

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